International
Journal
Of Advanced Research in Engineering & Management (IJAREM)

Harmonic Analysis Associated with the Generalized Dunkl-Bessel-Laplace Operator

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ABSTRACT : In this paper we consider a generalized Dunkl-Bessel-Laplace operator $\Delta_{k,\beta,n}$ which generalizes the Dunkl-Bessel-Laplace operator $\Delta_{k,\beta}$ on $\mathbb{R}^d\times]0,\infty[$, we define the generalized Dunkl-Bessel intertwining operator $\mathcal{R}_{k,\beta,n}$ and its dual ${}^t\mathcal{R}_{k,\beta,n}$. We exploit these operators to develop a new harmonic analysis corresponding to $\Delta_{k,\beta,n}$.

1 Introduction

In this paper we consider the generalized Dunkl-Bessel-Laplace operator defined by

$$\Delta_{k,\beta,n} = \Delta_{k,x'} + L_{\beta,n,x_{d+1}},\ x'\in\mathbb{R}^d,\ x_{d+1}>0, \tag{1}$$

where Δ_k is the Dunkl-Laplacian operator on $\mathbb{R}^d$ (see[2]), $L_{\beta,n}$ is the generalized Bessel operator on $]0,+\infty[$ given by

$$L_{\beta,n} = \frac{d^2}{dx_{d+1}^2} + \frac{2\beta+1}{x_{d+1}}\frac{d}{dx_{d+1}} - \frac{4n(\alpha+n)}{x_{d+1}^2},\ \beta > \frac{-1}{2}, \tag{2}$$

k is a multiplicity function (see [3]) and $n=0,1,\ldots$. For $n=0$, we regain the Dunkl-Bessel-Laplace operator.

$$\Delta_{k,\beta} = \Delta_{k,x'} + L_{\beta,x_{d+1}},\ x'\in\mathbb{R}^d,\ x_{d+1}>0. \tag{3}$$

Through this paper, we provide a new harmonic analysis on $\mathbb{R}^d\times]0,\infty[$ corresponding to the generalized Dunkl-Bessel-Laplace operator $\Delta_{k,\beta,n}$.

The outline of the content of this paper is as follows.

Section 2 is dedicated to some properties and results concerning the Dunkl-Laplace-Bessel operator .

In section 3, we construct the generalized Dunkl-Bessel intertwining operator $\mathcal{R}_{k,\beta,n}$ and its dual ${}^t\mathcal{R}_{k,\beta,n}$, next we exploit these operators to build a new harmonic analysis on $\mathbb{R}^d\times]0,\infty[$ corresponding to operator $\Delta_{k,\beta,n}$.

2 Preliminaries

Throughout this paper, we denote by

- $a_\beta = \dfrac{2\Gamma(\beta+1)}{\sqrt{\pi}\,\Gamma(\beta+\frac{1}{2})}$, where $\beta > \dfrac{-1}{2}$.
- $x=(x_1,\ldots,x_{d+1})=(x',x_{d+1})\in\mathbb{R}^d\times]0,\infty[$.
- $\lambda=(\lambda_1,\ldots,\lambda_{d+1})=(\lambda',\lambda_{d+1})\in\mathbb{C}^{d+1}$.
- $C(\mathbb{R}^{d+1})$ the space of continuous functions on $\mathbb{R}^{d+1}$, even with respect to the last variable.

- $E(\mathbb{R}^{d+1})$ (resp. $D(\mathbb{R}^{d+1})$) the space of C^{∞} functions on $\mathbb{R}^{d+1}$, even with respect to the last variable (resp. with compact support).
- $S(\mathbb{R}^{d+1})$ the Schwartz space of rapidly decreasing functions on $\mathbb{R}^{d+1}$ which are even with respect to the last variable.
- $\mathcal{R}$ the root system in $\mathbb{R}^d \setminus \{0\}$, $\mathcal{R}_+$ is a fixed positive subsystem and $k \in \mathcal{R} \to]0,\infty[$ a multiplicity function.
- T_j the Dunkl operator defined for $j = 1,...,d,$ on $\mathbb{R}^d$ and $f \in E(\mathbb{R}^d)$ by

$$T_j f(x) = \frac{df(x)}{dx_j} + \sum_{\alpha \in \mathcal{R}_+} k(\alpha)\alpha_j \frac{(f(x) - f(\sigma_\alpha(x))}{<\alpha, x>}$$

where $<,>$ is the usual scalar product, σ_α is the orthogonal reflection in the hyperplane orthogonal to α and the multiplicity function k is invariant by the finite reflection group W generated by the reflection σ_α $(\alpha \in \mathcal{R})$.

- Δ_k the Dunkl-Laplace operator defined by

$$\Delta_k f(x) = \sum_{j=0}^{d} T_j^2 f(x).$$

- w_k the weight function defined by

$$w_k(x') = \prod_{\alpha \in \mathcal{R}_+} |< \alpha, x' >|^{2k(\alpha)}, x' \in \mathbb{R}^d.$$

In this section we recall some facts about harmonic analysis related to the Dunkl-Bessel-Laplace operator $\Delta_{k,\beta}$. We cite here, as briefly as possible, only some properties. For more details we refer to [2, 3, 4].

Definition 1 *For all* $x \in \mathbb{R}^d \times]0,\infty[$ *we define the measure* $\xi_x^{k,\beta}$ *on* $\mathbb{R}^d \times]0,\infty[$ *by*

$$d\xi_x^{k,\beta}(y) = a_\beta x_{d+1}^{-2\beta}(x_{d+1}^2 - y_{d+1}^{d+1})^{\beta - \frac{1}{2}} 1_{]0,x_{d+1}[}(y_{d+1}) d\mu_{x'}(y') dy_{d+1},$$

where $\mu_{x'}$ is a probability measure on $\mathbb{R}^d$, with support in the closed ball $B(o, \|x\|)$ of center o and radius $\|x\|$. $1_{]0,x_{d+1}[}$ is the characteristic function of the interval $]0, x_{d+1}[$.

Definition 2 *The Dunkl-Bessel intertwining operator is the operator* $\mathcal{R}_{k,\beta}$ *defined on* $C(\mathbb{R}^{d+1})$ *by*

$$\mathcal{R}_{k,\beta} f(x', x_{d+1}) = a_\beta x_{d+1}^{-2\beta} \int_0^{x_{d+1}} (x_{d+1}^2 - t^2)^{\beta - \frac{1}{2}} V_k f(x', t) dt. \quad (4)$$

Remark 1 $\mathcal{R}_{k,\beta}$ *can also be written in the form*

$$\forall x \in \mathbb{R}^d \times [0,\infty[, \mathcal{R}_{k,\beta} f(x) = \int_{\mathbb{R}^d \times [0,\infty[} f(y) d\xi_x^{k,\beta}(y).$$

Proposition 1 $\mathcal{R}_{k,\beta}$ *is a topological isomorphism from* $E(\mathbb{R}^{d+1})$ *onto itself satisfying the following* *transmutation relation*

$$\Delta_{k,\beta}(\mathrm{R}_{k,\beta} f) = \mathrm{R}_{k,\beta}(\Delta_{d+1} f), \forall f \in E(\mathrm{R}^{d+1}),$$

where $\Delta_{d+1} = \sum_{j=1}^{d+1} \frac{d^2}{dx_j^2}$ is the Laplacian on R^{d+1}.

Definition 3 *The dual of the Dunkl-Bessel intertwining operator* $\mathrm{R}_{k,\beta}$ *is the operator* ${}^t\mathrm{R}_{k,\beta}$ *defined on* $D(\mathrm{R}^{d+1})$ *by:* $\forall y = (y', y_{d+1}) \in \mathrm{R}^d \times [0;\infty[$,

$$^t\mathrm{R}_{k,\beta}(f)(y', y_{d+1}) = a_\beta \int_{y_{d+1}}^{\infty} (s^2 - y_{d+1}^2)^{\beta-\frac{1}{2}}\, {}^tV_k f(y', s) s ds, \tag{5}$$

where ${}^tV_k$ is the dual Dunkl intertwining operator defined by

$$\forall y \in \mathrm{R}^d, {}^tV_k(y) = \int_{\mathrm{R}^d} f(x) d\nu_y(x), \tag{6}$$

and ν_y is a positive measure on R^d with support in the set $\{x \in \mathbb{R}^d, \|x\| \ge \|y\|\}$.

Proposition 2 ${}^t\mathrm{R}_{k,\beta}$ *is a topological isomorphism from* $S(\mathrm{R}^{d+1})$ *onto itself satisfying the following transmutation relation*

$$^t\mathrm{R}_{k,\beta}(\Delta_{k,\beta} f) = \Delta_{d+1}({}^t\mathrm{R}_{k,\beta} f), \forall f \in E(\mathrm{R}^{d+1}),$$

For all $y \in \mathrm{R}^d$, we define the measure $\rho_y^{k,\beta}$ on $\mathrm{R}^d \times [0,\infty[$, by

$$d\rho_y^{k,\beta}(x) = a_\beta (x_{d+1}^2 - y_{d+1}^2)^{\beta-\frac{1}{2}} x_{d+1} 1_{]y_{d+1},\infty[}(x_{d+1}) d\nu_{y'}(x') dx_{d+1}. \tag{7}$$

From (5) the operator ${}^t\mathrm{R}_{k,\beta}$ can also be written in the form

$$^t\mathrm{R}_{k,\beta}(f)(y) = \int_{\mathrm{R}^d \times [0,\infty[} f(x) d\rho_y^{k,\beta}(x). \tag{8}$$

We consider the function $\Lambda_{k,\beta}$, given for $\lambda = (\lambda', \lambda_{d+1}) \in \mathrm{C}^d \times \mathrm{C}$ by

$$\Lambda_{k,\beta}(x,\lambda) = K(x', -i\lambda') j_\beta(x_{d+1}\lambda_{d+1}), \tag{9}$$

where $j_\beta(x_{d+1}\lambda_{d+1})$ is the normalized Bessel function defined by

$$j_\beta(x_{d+1}\lambda_{d+1}) = a_\beta \int_0^1 (1-t^2)^{\beta-\frac{1}{2}} \cos(x_{d+1}\lambda_{d+1} t) dt$$

and $K(x', -i\lambda')$ is the Dunkl Kernel defined by

$$K(x', -i\lambda') = \int_{\mathrm{R}^d} e^{-i<y,\lambda'>} d\mu_{x'}(y).$$

The Dunkl-Bessel-Laplace operator $\Delta_{k,\beta,n}$ and the function $\Lambda_{k,\beta}$ are related by the following relation

$$\Delta_{k,\beta}(\Lambda_{k,\beta})(x,\lambda) = -\|\lambda\|^2 \Lambda_{k,\beta}(x,\lambda). \tag{10}$$

The Dunkl-Bessel translation operators T_x are defined by

$$T_x f(y) = \tau_{x'} \otimes T_{x_{d+1}}^\beta f(y', y_{d+1}),\ y' \in \mathrm{R}^d,\ y_{d+1} > 0 \tag{11}$$

where $\tau_{x'}$ is the Dunkl translation operator, and $T_{x_{d+1}}^\beta$ is the generalized translation operator associated with the Bessel operator L_β. We denote by $L^p_{k,\beta}(\mathbb{R}^d\times\mathbb{R}_+)$, $1\le p\le+\infty$ the space of measurable functions on $\mathbb{R}^d\times\mathbb{R}_+$ such that

$$\|f\|_{k,\beta,p}=\left(\int_{R^d\times R_+}|f(x)|^p\,\mathcal{A}_{k,\beta}(x)dx\right)^{\frac{1}{p}}<+\infty,\ if\ 1\le p<+\infty, \tag{12}$$

$$\|f\|_{k,\beta,\infty}=ess\sup_{x\in\mathbb{R}^d\times[0,+\infty[}|f(x)|<+\infty,\ if\ p=\infty \tag{13}$$

where

$$\mathcal{A}_{k,\beta}(x)dx=w_k(x')x_{d+1}^{2\beta+1}dx'dx_{d+1},\ x=(x',x_{d+1})\in\mathbb{R}^d\times\mathbb{R}^+. \tag{14}$$

Proposition 3 *Let* f *be in* $L^1_{k,\beta}(\mathbb{R}^d\times\mathbb{R}_+)$. *Then*

$$\int_{\mathbb{R}^d\times\mathbb{R}_+}{}^t\mathcal{R}_{k,\beta}(f)(y)dy=\int_{\mathbb{R}^d\times\mathbb{R}_+}f(x)\mathcal{A}_{k,\beta}(x)dx.$$

Theorem 1 *Let* $f\in L^1_{k,\beta}(\mathbb{R}^d\times\mathbb{R}_+)$ *and* g *in* $C(\mathbb{R}^{d+1})$, *we have the formula*

$$\int_{\mathbb{R}^d\times\mathbb{R}_+}{}^t\mathcal{R}_{k,\beta}(f)(y)g(y)dy=\int_{\mathbb{R}^d\times\mathbb{R}_+}f(x)\mathcal{R}_{k,\beta}(g)(x)\mathcal{A}_{k,\beta}(x)dx. \tag{15}$$

Definition 4 *The Dunkl-Bessel transform is given for* f *in* $D(\mathbb{R}^{d+1})$ *by*

$$\forall\lambda\in\mathbb{R}^d\times\mathbb{R}_+,\ \mathcal{F}_{k,\beta}(f)(\lambda)=\int_{\mathbb{R}^d\times\mathbb{R}_+}f(x)\Lambda_{k,\beta}(x,\lambda)\mathcal{A}_{k,\beta}(x)dx. \tag{16}$$

Remark 2 *The relation (16) can also be written in the following form:*

$$\forall\lambda=(\lambda',\lambda_{d+1})\in\mathbb{R}^d\times\mathbb{R}_+,\ \mathcal{F}_{k,\beta}(f)(\lambda)=\mathcal{F}_k\circ\mathcal{F}_\beta(f)(\lambda), \tag{17}$$

where $\mathcal{F}_k$ is the Dunkl transform of a function ψ in $\mathbb{R}^d$ given by

$$\forall\lambda'\in\mathbb{R}^d,\ \mathcal{F}_k(\psi)(\lambda')=\int_{\mathbb{R}^d}\psi(x')K(x',-i\lambda')w_k(x')dx' \tag{18}$$

and $\mathcal{F}_\beta$ is the Fourier-Bessel transform defined for $h\in D(\mathbb{R})$ by

$$\forall\lambda_{d+1}\in\mathbb{R}_+,\ \mathcal{F}_\beta(\lambda_{d+1})=\int_{\mathbb{R}_+}h(t)j_\beta(\lambda_{d+1}t)t^{2\beta+1}dt. \tag{19}$$

Proposition 4

- For $f\in L^1_{k,\beta}(\mathbb{R}^d\times\mathbb{R}_+)$, we have

$$\|\ F_{k,\beta}(f)\ \|_{k,\beta,\infty}\le\|\ f\ \|_{k,\beta,1}. \tag{20}$$

- For $f\in D(\mathbb{R}^{d+1})$, we have

$$\mathcal{F}_{k,\beta}(f)=\mathcal{F}_0\circ {}^t\mathcal{R}_{k,\beta}(f), \tag{21}$$

where $\mathcal{F}_0$ is the transform defined by $\forall\lambda=(\lambda',\lambda_{d+1})\in\mathbb{R}^d\times\mathbb{R}_+$

$$\mathcal{F}_0(f)(\lambda',\lambda_{d+1})=\int_{\mathbb{R}^d\times\mathbb{R}_+} f(x',x_{d+1})e^{-i<\lambda',x'>}\cos(x_{d+1}\lambda_{d+1})dx'dx_{d+1}. \tag{22}$$

• For $f\in D(\mathbb{R}^{d+1})$, we have

$$\forall\lambda\in R^d\times R_+,\ F_{k,\beta}(\Delta_{k,\beta}f)(\lambda)=-\|\lambda\|^2 F_{k,\beta}(f)(\lambda). \tag{23}$$

Theorem 2 *The inverse transform* $\mathcal{F}_{k,\beta}^{-1}$ *is given by*

$$\forall\lambda\in\mathbb{R}^d\times\mathbb{R}_+,\ \mathcal{F}_{k,\beta}^{-1}(f)(y)=m_{k,\beta}\mathcal{F}_{k,\beta}(f)(-y) \tag{24}$$

with

$$m_{k,\beta}=\frac{c_k^2}{4^{\gamma+\beta+d}(\Gamma(\beta+1))^2} \tag{25}$$

and c_k is given by Mehta integral

$$\frac{1}{c_k}=\int_{\mathbb{R}^d}\exp(-\|x\|^2)w_k dx$$

and

$$\gamma=\sum_{\alpha\in\mathcal{R}_+}k(\alpha). \tag{26}$$

Theorem 3 *For all* $f\in L^1_{k,\beta}(\mathbb{R}^d\times\mathbb{R}_+)$ *such that* $\mathcal{F}_{k,\beta}(f)\in L^1_{k,\beta}(\mathbb{R}^d\times\mathbb{R}_+)$, *we have the inverse formula*

$$f(y)=m_{k,\beta}\int_{\mathbb{R}^d\times\mathbb{R}_+}\mathcal{F}_{k,\beta}(f)(\lambda)\Lambda_{k,\beta}(-y,\lambda)\mathcal{A}_{k,\beta}(\lambda)d\lambda,\ a.e. \tag{27}$$

Theorem 4 *Plancherel formula: for all* f *in* $D(\mathbb{R}^{d+1})$, *we have*

$$\int_{\mathbb{R}^d\times\mathbb{R}_+}|f(x)|^2\mathcal{A}_{k,\beta}(x)dx=m_{k,\beta}\int_{\mathbb{R}^d\times\mathbb{R}_+}|\mathcal{F}_{k,\beta}(f)(\lambda)|^2\mathcal{A}_{k,\beta}(\lambda)d\lambda. \tag{28}$$

Definition 5 *The translation operators* $T_x,\ x\in\mathbb{R}^d\times\mathbb{R}_+$, *associated with the Dunkl-Bessel operator are defined for* $f\in L^2_{k,\beta}(\mathbb{R}^d\times\mathbb{R}_+)$ *and* $\lambda\in\mathbb{R}^d\times\mathbb{R}_+$ *by*

$$\mathcal{F}_{k,\beta}(T_x f)(\lambda)=\Lambda_{k,\beta}(x,\lambda)\mathcal{F}_{k,\beta}(f)(\lambda). \tag{29}$$

Proposition 5

- For all $x, y \in \mathbb{R}^d \times \mathbb{R}_+$ and $\lambda \in \mathbb{C}^{d+1}$, we have

$$T_x \Lambda_{k,\beta}(y,\lambda) = \Lambda_{k,\beta}(x,\lambda)\Lambda_{k,\beta}(y,\lambda). \tag{30}$$

- For $f \in E(\mathbb{R}^{d+1})$, and $g \in D(\mathbb{R}^{d+1})$, we have

$$\int_{\mathbb{R}^d \times \mathbb{R}_+} T_x f(y) g(y) A_{k,\beta}(y) dy = \int_{\mathbb{R}^d \times \mathbb{R}_+} f(y) T_x g(y) A_{k,\beta}(y) dy. \tag{31}$$

Definition 6 *The convolution product associated with the Dunkl-Bessel operator of two functions* f *and* g *in* $D(\mathbb{R}^{d+1})$ *is defined on* $\mathbb{R}^d \times \mathbb{R}_+$ *by*

$$f *_{k,\beta} g(x) = \int_{\mathbb{R}^d \times \mathbb{R}_+} f(y) T_x g(y^-) A_{k,\beta}(y) dy, \tag{32}$$

with $y^- = (-y', y_{d+1})$.

Proposition 6 *Let* $f \in L^2_{k,\beta}(\mathbb{R}^{d+1})$ *and* $g \in L^1_{k,\beta}(\mathbb{R}^{d+1})$, *we have*

$$\mathcal{F}_{k,\beta}(f *_{k,\beta} g) = \mathcal{F}_{k,\beta}(f)\mathcal{F}_{k,\beta}(g). \tag{33}$$

3 Harmonic analysis associated with $\Delta_{k,\beta,n}$

Throughout this section we denoted by

- $\mathbb{R}^{d+1}_+ = \mathbb{R}^d \times]0,\infty[$.
- $\mathcal{M}_n$ the map defined by $\mathcal{M}_n f(x', x_{d+1}) = x^{2n}_{d+1} f(x', x_{d+1})$.
- $L^p_{k,\beta,n}(\mathbb{R}^{d+1}_+)$ the class of measurable functions f on $\mathbb{R}^{d+1}_+$ for which

$$\| f \|_{k,\beta,n,p} = \| M_n^{-1} f \|_{k,\beta+2n,p} < \infty.$$

- $E_n(\mathbb{R}^{d+1})$ (resp. $D_n(\mathbb{R}^{d+1})$ and $S_n(\mathbb{R}^{d+1})$) stand for the subspace of $E(\mathbb{R}^{d+1})$ (resp. $D(\mathbb{R}^{d+1})$ and $S(\mathbb{R}^{d+1})$) consisting of functions f such that

$$f(x',0) = \left(\frac{d^k f}{dx^k_{d+1}}\right)(x',0) = 0, \forall k \in \{1,...2n-1\}.$$

We consider the function $\Lambda_{k,\beta,n}$, given for $\lambda = (\lambda', \lambda_{d+1}) \in \mathbb{C}^d \times \mathbb{C}$ by

$$\Lambda_{k,\beta,n}(x,\lambda) = x^{2n}_{d+1}\Lambda_{k,\beta+2n}(x,\lambda) = \mathcal{M}_n \Lambda_{k,\beta+2n}(x,\lambda). \tag{34}$$

Lemma 1

- The map $\mathcal{M}_n$ is an isomorphism
  - from $E(\mathbb{R}^{d+1})$ onto $E_n(\mathbb{R}^{d+1})$.

- from $S(\mathbb{R}^{d+1})$ onto $S_n(\mathbb{R}^{d+1})$.

• For all $f \in E(\mathbb{R})$ we have

$$L_{\beta,n} \circ \mathrm{M}_n(f) = \mathrm{M}_n \circ L_{\beta+2n}(f), \tag{35}$$

where $L_{\beta,n}$ is the generalized Bessel operator given by (2).

• For all $f \in E(\mathbb{R}^{d+1})$

$$\Delta_{k,\beta,n} \circ \mathrm{M}_n(f)(x', x_{d+1}) = \mathrm{M}_n \circ \Delta_{k,\beta+2n}(f)(x', x_{d+1}). \tag{36}$$

Proof. For assertion (i) and (ii) (See [1]). For assertion (iii), using (1) and (35), we have for any $f \in E(\mathbb{R}^{d+1})$

$$\begin{aligned}
&\Delta_{k,\beta,n} \circ \mathrm{M}_n f(x', x_{d+1}) = \Delta_{k,x'} \circ \mathrm{M}_n f(x', x_{d+1}) + L_{\beta,n,x_{d+1}} \circ \mathrm{M}_n f(x', x_{d+1}) \\
&= \Delta_{k,x'} x_{d+1}^{2n} f(x', x_{d+1}) + L_{\beta,n,x_{d+1}} x_{d+1}^{2n} f(x', x_{d+1}) \\
&= x_{d+1}^{2n} \Delta_{k,x'} f(x', x_{d+1}) + x_{d+1}^{2n} L_{\beta+2n,x_{d+1}} f(x', x_{d+1}) \\
&= \mathrm{M}_n \circ \Delta_{k,\beta+2n}(f)(x', x_{d+1}),
\end{aligned}$$

where $x' \in \mathbb{R}^d$ and $x_{d+1} > 0$.

Proposition 7 *The function* $\Lambda_{k,\beta,n}$ *satisfies the differential equation*

$$\Delta_{k,\beta,n}(\Lambda_{k,\beta,n})(x,\lambda) = -\| \ \lambda \ \|^2 \Lambda_{k,\beta,n}(x,\lambda).$$

Proof. From (34) we have

$$\Lambda_{k,\beta,n} = \mathrm{M}_n \circ \Lambda_{k,\beta+2n},$$

using (10) and (36) we obtain

$$\begin{aligned}
&\Delta_{k,\beta,n}(\Lambda_{k,\beta,n}) = \Delta_{k,\beta,n}(\mathrm{M}_n \Lambda_{k,\beta+2n}) \\
&= \mathrm{M}_n \Delta_{k,\beta+2n}(\Lambda_{k,\beta+2n}) \\
&= -\| \ \lambda \ \|^2 M_n \Lambda_{k,\beta+2n} \\
&= -\| \ \lambda \ \|^2 \Lambda_{k,\beta,n}(x,\lambda).
\end{aligned}$$

Definition 7 *The generalized Dunkl-Bessel intertwining operator is the operator* $\mathrm{R}_{k,\beta,n}$ *defined on* $C(\mathbb{R}^{d+1})$ *by*

$$\mathrm{R}_{k,\beta,n} f(x) = a_{\beta+2n} x_{d+1}^{-2(\beta+n)} \int_0^{x_{d+1}} (x_{d+1}^2 - t^2)^{\beta+2n-\frac{1}{2}} V_k f(x', t) dt.$$

Remark 3

• From (4) it is easily checked that

$$\mathrm{R}_{k,\beta,n} = \mathrm{M}_n \circ \mathrm{R}_{k,\beta+2n}. \tag{37}$$

• From Definition 1, Remark 1 and (37) $\mathrm{R}_{k,\beta,n}$ can also be written in the form

$$\forall x \in \mathbb{R}^{d+1}, \mathrm{R}_{k,\beta,n} f(x) = \int_{\mathbb{R}^{d+1}} x_{d+1}^{2n} f(y) d\xi_x^{k,\beta+2n}(y).$$

Proposition 8 $\mathrm{R}_{k,\beta,n}$ *is a topological isomorphism from* $E(\mathbb{R}^{d+1})$ *onto* $E_n(\mathbb{R}^{d+1})$ *satisfying the following transmutation relation*

$$\Delta_{k,\beta,n}(\mathrm{R}_{k,\beta,n} f) = \mathrm{R}_{k,\beta,n}(\Delta_{d+1} f), \forall f \in E(\mathbb{R}^{d+1}),$$

where $\Delta_{d+1} = \sum_{j=1}^{d+1} \frac{d^2}{dx_j^2}$ is the Laplacian on $\mathbb{R}^{d+1}$.

Proof. The result follows directly from (37), Proposition 1 and Lemma 1.((i) and (iii)).

Definition 8 *The dual of the generalized Dunkl-Bessel intertwining operator* $\mathrm{R}_{k,\beta,n}$ *is the operator defined on* $D_n(\mathbb{R}^{d+1})$ *by:* $\forall y = (y', y_{d+1}) \in \mathbb{R}^d \times]0;\infty[$,

$$ {}^t\mathrm{R}_{k,\beta,n}(f)(y', y_{d+1}) = a_{\beta+2n} \int_{y_{d+1}}^{\infty} (s^2 - y_{d+1}^2)^{\beta+2n-\frac{1}{2}} \, {}^tV_k f(y', s) s^{1-2n} ds. \tag{38}$$

Remark 4

- Due to (5) and (38)

$$ {}^t\mathrm{R}_{k,\beta,n} = {}^t\mathrm{R}_{k,\beta+2n} \circ \mathrm{M}_n^{-1}. \tag{39}$$

- By (7), (8) and (39) we can deduce that

$$ {}^t\mathrm{R}_{k,\beta,n}(f)(y) = \int_{\mathbb{R}_+^{d+1}} x_{d+1}^{-2n} f(x) d\rho_y^{k,\beta+2n}(x)$$

where $x = (x', x_{d+1}) \in \mathbb{R}^d \times]0,\infty[$.

Proposition 9 ${}^t\mathrm{R}_{k,\beta}$ *is a topological isomorphism from* $S_n(\mathbb{R}^{d+1})$ *onto* $S(\mathbb{R}^{d+1})$ *satisfying the following transmutation relation*

$$ {}^t\mathrm{R}_{k,\beta}(\Delta_{k,\beta} f) = \Delta_{d+1}({}^t\mathrm{R}_{k,\beta} f), \forall f \in S_n(\mathbb{R}^{d+1}).$$

Proof. The result follows directly from (39), Proposition 2 and Lemma 1.((i) and (iii)).

Proposition 10 *Let* f *be in* $L^1_{k,\beta,n}(\mathbb{R}_+^{d+1})$. *Then*

$$\int_{\mathbb{R}_+^{d+1}} {}^t\mathrm{R}_{k,\beta,n}(f)(y) dy = \int_{\mathbb{R}_+^{d+1}} f(x) \mathrm{A}_{k,\beta+n}(x) dx.$$

Proof. An easily combination of (14), (39) and Proposition 1 shows that

$$\begin{aligned} \int_{\mathbb{R}_+^{d+1}} {}^t\mathrm{R}_{k,\beta,n}(f)(y) dy &= \int_{\mathbb{R}_+^{d+1}} {}^t\mathrm{R}_{k,\beta+2n} \mathrm{M}_n^{-1}(f)(y) dy \\ &= \int_{\mathbb{R}_+^{d+1}} \mathrm{M}_n^{-1}(f)(x) \mathrm{A}_{k,\beta+2n}(x) dx \\ &= \int_{\mathbb{R}_+^{d+1}} x_{d+1}^{-2n} f(x) \mathrm{A}_{k,\beta+2n}(x) dx \end{aligned}$$

$$= \int_{\mathbb{R}_+^{d+1}} f(x) A_{k,\beta+n}(x)dx.$$

Theorem 5 *Let* $f \in L^1_{k,\beta,n}(\mathbb{R}_+^{d+1})$ *and* $g \in C(\mathbb{R}^{d+1})$, *we have the following formula*

$$\int_{\mathbb{R}_+^{d+1}} {}^t\mathcal{R}_{k,\beta,n}(f)(y)g(y)dy = \int_{\mathbb{R}_+^{d+1}} f(x)\mathcal{R}_{k,\beta,n}(g)(x)A_{k,\beta+n}(x)dx.$$

Proof. From (15) and (39) we obtain

$$\int_{\mathbb{R}_+^{d+1}} {}^t\mathcal{R}_{k,\beta,n}(f)(y)g(y)dy = \int_{\mathbb{R}_+^{d+1}} {}^t\mathcal{R}_{k,\beta+2n} \circ M_n^{-1}(f)(y)g(y)dy$$

$$= \int_{\mathbb{R}_+^{d+1}} M_n^{-1}(f)(x)\mathcal{R}_{k,\beta+2n}(g)(x)A_{k,\beta+2n}(x)dx$$

$$= \int_{\mathbb{R}_+^{d+1}} x_{d+1}^{-2n} f(x)\mathcal{R}_{k,\beta+2n}(g)(x)A_{k,\beta+2n}(x)dx$$

$$= \int_{\mathbb{R}_+^{d+1}} f(x)\mathcal{R}_{k,\beta,n}(g)(x)A_{k,\beta+n}(x)dx.$$

Definition 9 *The generalized Dunkl-Bessel transform is given for* f *in* $D_n(\mathbb{R}^{d+1})$ *by*

$$\forall \lambda \in \mathbb{R}^d \times \mathbb{R}_+,\ \mathcal{F}_{k,\beta,n}(f)(\lambda) = \int_{\mathbb{R}^d \times \mathbb{R}_+} f(x)\Lambda_{k,\beta,n}(x,\lambda)A_{k,\beta}(x)dx. \quad (40)$$

Remark 5

- Due to (14), (16) and (34) we have

$$\mathcal{F}_{k,\beta,n} = \mathcal{F}_{k,\beta+2n} \circ M_n^{-1}. \quad (41)$$

- By (17) and (41) we can deduce that

$$\mathcal{F}_{k,\beta,n} = \mathcal{F}_k\left(\mathcal{F}_{\beta+2n} \circ M_n^{-1}\right)$$

Proposition 11

- For $f \in L^1_{k,\beta,n}(\mathbb{R}^d \times \mathbb{R}_+)$, we have

$$\| F_{k,\beta,n}(f) \|_{k,\beta,\infty} \leq \| f \|_{k,\beta,n,1}.$$

- For $f \in D_n(\mathbb{R}^{d+1})$, we have

$$\mathcal{F}_{k,\beta,n}(f) = \mathcal{F}_0 \circ {}^t\mathcal{R}_{k,\beta,n}(f),$$

where $\mathcal{F}_0$ is the transform defined by $\forall \lambda = (\lambda', \lambda_{d+1}) \in \mathbb{R}^d \times \mathbb{R}_+$

$$\mathcal{F}_0(f)(\lambda', \lambda_{d+1}) = \int_{\mathbb{R}^d \times \mathbb{R}_+} f(x', x_{d+1})e^{-i<\lambda', x_{d+1}>}\cos(x_{d+1}\lambda_{d+1})dx'dx_{d+1}.$$

- For $f \in D(\mathbb{R}^{d+1})$, we have

$$\forall \lambda \in R^d \times R_+,\ F_{k,\beta,n}(\Delta_{k,\beta,n} f)(\lambda) = -\| \ \lambda \ \|^2 F_{k,\beta,n}(f)(\lambda).$$

Proof. From (20) and (41) we have

$$\begin{aligned}&\| \ F_{k,\beta,n}(f) \ \|_{k,\beta,\infty} = \| \ F_{k,\beta+2n} \circ M_n^{-1}(f) \ \|_{k,\beta,\infty}\\ &\leq \left\| M_n^{-1} f \right\|_{k,\beta+2n,1}\\ &\leq \| f \|_{k,\beta,n,1}.\end{aligned}$$

which proves assertion (i).

By (21), (39) and (41) we obtain

$$\begin{aligned}&\mathrm{F}_{k,\beta,n}(f) = \mathrm{F}_{k,\beta+2n} \circ \mathrm{M}_n^{-1}(f)\\ &= \mathrm{F}_0 \circ {}^t\mathrm{R}_{k,\beta+2n} \circ \mathrm{M}_n^{-1}(f)\\ &= \mathrm{F}_0 \circ {}^t\mathrm{R}_{k,\beta,n}(f),\end{aligned}$$

which proves assertion (ii).

Due to (23), (36) and (41) we have

$$\begin{aligned}&\mathrm{F}_{k,\beta,n}(\Delta_{k,\beta,n} f)(\lambda) = \mathrm{F}_{k,\beta+2n} \circ \mathrm{M}_n^{-1}(\Delta_{k,\beta,n} f)(\lambda)\\ &= \mathrm{F}_{k,\beta+2n} \circ \mathrm{M}_n^{-1}(\Delta_{k,\beta,n} f)(\lambda)\\ &= \mathrm{F}_{k,\beta+2n}(\Delta_{k,\beta+2n} \mathrm{M}_n^{-1} f)(\lambda)\\ &= -\|\lambda\|^2 F_{k,\beta+2n} \circ M_n^{-1}(f)(\lambda)\\ &= -\|\lambda\|^2 F_{k,\beta,n}(f)(\lambda).\end{aligned}$$

Theorem 6 *The inverse of the Dunkl-Bessel transform* $\mathrm{F}_{k,\beta,n}$ *is given by*

$$\forall y \in \mathrm{R}^d \times \mathrm{R}_+,\ \mathrm{F}_{k,\beta,n}^{-1}(f)(y) = m_{k,\beta+2n} \mathrm{F}_{k,\beta,n}(f)(-y). \tag{42}$$

Proof. By (24) and (41) we have

$$\begin{aligned}&\mathrm{F}_{k,\beta,n}(f)(y) = \mathrm{F}_{k,\beta+2n} \circ \mathrm{M}_n^{-1}(f)(y)\\ &\mathrm{F}_{k,\beta,n}^{-1}(f)(y) = \mathrm{M}_n \circ \mathrm{F}_{k,\beta+2n}^{-1}(f)(y)\\ &= m_{k,\beta+2n} \mathrm{M}_n \circ \mathrm{F}_{k,\beta+2n}(f)(-y)\\ &= m_{k,\beta+2n} \mathrm{F}_{k,\beta,n}(f)(-y).\end{aligned}$$

Theorem 7 *For all* $f \in L^1_{k,\beta,n}(\mathrm{R}_+^{d+1})$ *such that* $\mathrm{F}_{k,\beta,n}(f) \in L^1_{k,\beta,n}(\mathrm{R}_+^{d+1})$, *we have the inverse formula*

$$f(y) = m_{k,\beta+2n} \int_{\mathrm{R}^d \times \mathrm{R}_+} \mathrm{F}_{k,\beta,n}(f)(\lambda) \Lambda_{k,\beta,n}(-y,\lambda) \mathrm{A}_{k,\beta+2n}(\lambda) d\lambda,\ a.e. \tag{43}$$

Proof. An easily combination of (14), (27), (34) and (41) shows that

$$m_{k,\beta+2n} \int_{\mathrm{R}_+^{d+1}} \mathrm{F}_{k,\beta,n}(f)(\lambda) \Lambda_{k,\beta,n}(-y,\lambda) \mathrm{A}_{k,\beta+2n}(\lambda) d\lambda$$

$$= m_{k,\beta+2n}\int_{\mathbb{R}_+^{d+1}} \mathcal{F}_{k,\beta+2n}\circ \mathcal{M}_n^{-1}(f)(\lambda) y_{d+1}^{2n}\Lambda_{k,\beta+2n}(-y,\lambda)\mathcal{A}_{k,\beta+2n}(\lambda)d\lambda$$
$$= y_{d+1}^{2n}\mathcal{M}_n^{-1} f(y)$$
$$= f(y).$$

Theorem 8 *Plancherel formula: for all* f *in* $D_n(\mathbb{R}^{d+1})$, *we have*

$$\int_{\mathbb{R}_+^{d+1}} | f(x)|^2 \mathcal{A}_{k,\beta}(x)dx = m_{k,\beta+2n}\int_{\mathbb{R}_+^{d+1}} |\mathcal{F}_{k,\beta,n}(f)(\lambda)|^2 \mathcal{A}_{k,\beta+2n}(\lambda)d\lambda.$$

Proof. By (41) we have

$$\int_{\mathbb{R}_+^{d+1}} |\mathcal{F}_{k,\beta,n}(f)(\lambda)|^2 \mathcal{A}_{k,\beta+2n}(\lambda)d\lambda = \int_{\mathbb{R}_+^{d+1}} |\mathcal{F}_{k,\beta+2n}\circ\mathcal{M}_n^{-1}(f)(\lambda)|^2 \mathcal{A}_{k,\beta+2n}(\lambda)d\lambda,$$

using (28) we get

$$m_{k,\beta+2n}\int_{\mathbb{R}_+^{d+1}} |\mathcal{F}_{k,\beta+2n}\circ\mathcal{M}_n^{-1}(f)(\lambda)|^2 \mathcal{A}_{k,\beta+2n}(\lambda)d\lambda = \int_{\mathbb{R}_+^{d+1}} |\mathcal{M}_n^{-1} f(x)|^2 \mathcal{A}_{k,\beta+2n}(x)dx$$
$$= \int_{\mathbb{R}_+^{d+1}} | f(x)|^2 \mathcal{A}_{k,\beta}(x)dx.$$

Definition 10 *The generalized Dunkl-Bessel translation operators* $\mathcal{T}_x$ *associated with* $\Delta_{k,\beta,n}$ *are defined by*

$$\mathcal{T}_x = x_{d+1}^{2n}\mathcal{M}_n\circ T_x\circ\mathcal{M}_n^{-1}. \tag{44}$$

Remark 6 *By (11) and (44) it is easily checked that*

$$\mathcal{T}_x f(y) = \tau_{x'}\otimes T_{x_{d+1}}^{\beta,n} f(y', y_{d+1})$$

where $T_{x_{d+1}}^{\beta,n}$ are the generalized Bessel translation operators defined by

$$T_{x_{d+1}}^{\beta,n} = x_{d+1}^{2n}\mathcal{M}_n\circ T_{x_{d+1}}^{\beta+2n}\circ\mathcal{M}_n^{-1}.$$

Proposition 12 *Let* $f\in L^2_{k,\beta}(\mathbb{R}_+^{d+1})$, $x\in\mathbb{R}_+^{d+1}$ *and* $\lambda\in\mathbb{R}_+^{d+1}$ *then*

$$\mathcal{F}_{k,\beta,n}(\mathcal{T}_x f)(\lambda) = \Lambda_{k,\beta,n}(x,\lambda)\mathcal{F}_{k,\beta,n}(f)(\lambda).$$

Proof. Using (29), (34), (41) and (44) we get

$$\mathcal{F}_{k,\beta,n}(\mathcal{T}_x f)(\lambda) = \mathcal{F}_{k,\beta+2n}\circ\mathcal{M}_n^{-1}(x_{d+1}^{2n}\mathcal{M}_n\circ T_x\circ\mathcal{M}_n^{-1}(f))(\lambda)$$
$$= \mathcal{F}_{k,\beta+2n}\circ\mathcal{M}_n^{-1}(x_{d+1}^{2n}\mathcal{M}_n\circ T_x\circ\mathcal{M}_n^{-1}(f))(\lambda)$$
$$= x_{d+1}^{2n}\mathcal{F}_{k,\beta+2n}(T_x\circ\mathcal{M}_n^{-1}(f))(\lambda)$$
$$= x_{d+1}^{2n}\Lambda_{k,\beta+2n}(x,\lambda)\mathcal{F}_{k,\beta+2n}\circ\mathcal{M}_n^{-1}(f)(\lambda)$$
$$= \Lambda_{k,\beta,n}(x,\lambda)\mathcal{F}_{k,\beta,n}(f)(\lambda).$$

Proposition 13

- For all $x, y\in\mathbb{R}_+^{d+1}$ and $\lambda\in\mathbb{C}^{d+1}$, we have

$$\mathrm{T}_x\Lambda_{k,\beta,n}(y,\lambda)=\Lambda_{k,\beta,n}(x,\lambda)\Lambda_{k,\beta,n}(y,\lambda).$$

• For $f\in E(\mathbb{R}^{d+1})$, and $g\in D(\mathbb{R}^{d+1})$, we have

$$\int_{\mathbb{R}_+^{d+1}}T_x f(y)g(y)A_{k,\beta}(y)dy=\int_{\mathbb{R}_+^{d+1}}f(y)T_x g(y)A_{k,\beta}(y)dy.$$

Proof. An easily combination of (30), (34) and (44) shows that

$$\mathrm{T}_x\Lambda_{k,\beta,n}(y,\lambda)=x_{d+1}^{2n}\mathrm{M}_n\circ T_x\circ \mathrm{M}_n^{-1}\Lambda_{k,\beta,n}(y,\lambda)$$
$$=x_{d+1}^{2n}\mathrm{M}_n\circ T_x\Lambda_{k,\beta+2n}(y,\lambda)$$
$$=x_{d+1}^{2n}y_{d+1}^{2n}T_x\Lambda_{k,\beta+2n}(y,\lambda)$$
$$=x_{d+1}^{2n}y_{d+1}^{2n}\Lambda_{k,\beta+2n}(x,\lambda)\Lambda_{k,\beta+2n}(y,\lambda)$$
$$=\Lambda_{k,\beta,n}(x,\lambda)\Lambda_{k,\beta,n}(y,\lambda).$$

which proves assertion (i).
By (14), (31) and (44) we obtain

$$\int_{\mathbb{R}_+^{d+1}}\mathrm{T}_x f(y)g(y)A_{k,\beta}(y)dy=\int_{\mathbb{R}_+^{d+1}}x_{d+1}^{2n}\mathrm{M}_n\circ T_x\circ \mathrm{M}_n^{-1}f(y)g(y)A_{k,\beta}(y)dy$$
$$=\int_{\mathbb{R}_+^{d+1}}x_{d+1}^{2n}T_x\circ \mathrm{M}_n^{-1}f(y)\mathrm{M}_n^{-1}g(y)A_{k,\beta+2n}(y)dy$$
$$=\int_{\mathbb{R}_+^{d+1}}\mathrm{M}_n^{-1}f(y)x_{d+1}^{2n}T_x\circ \mathrm{M}_n^{-1}g(y)A_{k,\beta+2n}(y)dy$$
$$=\int_{\mathbb{R}_+^{d+1}}f(y)x_{d+1}^{2n}\mathrm{M}_n\circ T_x\circ \mathrm{M}_n^{-1}g(y)A_{k,\beta}(y)dy$$
$$=\int_{\mathbb{R}_+^{d+1}}f(y)\mathrm{T}_x g(y)A_{k,\beta}(y)dy.$$

Definition 11 *The convolution product associated with the generalized Dunkl-Bessel operator of two functions* f *and* g *in* $D_n(\mathbb{R}^{d+1})$ *is defined on* $\mathbb{R}_+^{d+1}$ *by*

$$f *_{k,\beta,n} g(x)=\int_{\mathbb{R}_+^{d+1}}f(y)\mathrm{T}_x g(y^-)A_{k,\beta}(y)dy, \tag{45}$$

with $y^-=(-y',y_{d+1})$.

Proposition 14 *Let* f *and* g *in* $D_n(\mathbb{R}^{d+1})$, *we have*

$$f *_{k,\beta,n} g=\mathrm{M}_n\left[(\mathrm{M}_n^{-1}f)*_{k,\beta+2n}(\mathrm{M}_n^{-1}g)\right] \tag{46}$$

Proof. From (32), (44) and (45) we obtain

$$f *_{k,\beta,n} g(x)=\int_{\mathbb{R}_+^{d+1}}f(y)\mathrm{T}_x g(y^-)A_{k,\beta}(y)dy$$
$$=\int_{\mathbb{R}_+^{d+1}}f(y)x_{d+1}^{2n}\mathrm{M}_n\circ T_x\circ \mathrm{M}_n^{-1}g(y^-)A_{k,\beta}(y)dy$$
$$=x_{d+1}^{2n}\int_{\mathbb{R}_+^{d+1}}\mathrm{M}_n^{-1}f(y)T_x\circ \mathrm{M}_n^{-1}g(y^-)A_{k,\beta+2n}(y)dy$$
$$=\mathrm{M}_n\left[(\mathrm{M}_n^{-1}f)*_{k,\beta+2n}(\mathrm{M}_n^{-1}g)\right]$$

Proposition 15 *Let* $f \in L^2_{k,\beta,n}(\mathbb{R}^{d+1}_+)$ *and* $g \in L^1_{k,\beta,n}(\mathbb{R}^{d+1}_+)$, *then*

$$\mathcal{F}_{k,\beta,n}(f *_{k,\beta,n} g) = \mathcal{F}_{k,\beta,n}(f)\mathcal{F}_{k,\beta,n}(g). \tag{47}$$

Proof. By (46) we have

$$f *_{k,\beta,n} g = \mathcal{M}_n\left[(\mathcal{M}_n^{-1} f) *_{k,\beta+2n} (\mathcal{M}_n^{-1} g)\right]$$

using (33) and (41) we get

$$\begin{aligned}
&\mathcal{F}_{k,\beta,n}(f *_{k,\beta,n} g) = \mathcal{F}_{k,\beta,n} \circ \mathcal{M}_n\left[(\mathcal{M}_n^{-1} f) *_{k,\beta+2n} (\mathcal{M}_n^{-1} g)\right] \\
&= \mathcal{F}_{k,\beta+2n} \circ \mathcal{M}_n^{-1} \circ \mathcal{M}_n\left[(\mathcal{M}_n^{-1} f) *_{k,\beta+2n} (\mathcal{M}_n^{-1} g)\right] \\
&= \mathcal{F}_{k,\beta+2n}\left[(\mathcal{M}_n^{-1} f) *_{k,\beta+2n} (\mathcal{M}_n^{-1} g)\right] \\
&= \mathcal{F}_{k,\beta+2n}(\mathcal{M}_n^{-1} f)\mathcal{F}_{k,\beta+2n}(\mathcal{M}_n^{-1} g) \\
&= \mathcal{F}_{k,\beta,n}(f)\mathcal{F}_{k,\beta,n}(g).
\end{aligned}$$

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