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Analytical similarity solution of non-linear equation using one-parameter infinitesimal Lie group of transformation

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Abstract: The present paper discusses the analytical solution of the equation of flow of water through a porous medium of an aquifer. The flow in the aquifer is assumed to be one-dimensional and unconfined. Also the flow is considered to be unsteady and saturated. The equation governing such flow is a non-linear partial differential equation along with a source term. This non-linear partial differential equation is reduced to ordinary differential equation using the Lie- infinitesimal transformation technique to find similarity solutions. The analytical solution is obtained for the ordinary differential equation which is Abel's equation of second kind.

Keywords: Aquifer, Unsteady flow, unconfined flow, Similarity solution

Introduction

An aquifer is a body of saturated rock through which water can easily move. It is a layer of porous substrate that contains and transmits groundwater. Water squeezes through pore spaces of rock and sediment to move through an aquifer. Underground water moves very rapidly in fractured rock aquifers. There are three different types of aquifers-confined, unconfined and perched. When water flows directly between the surface and the saturated zone of an aquifer then the aquifers are referred as unconfined.

In this paper, we have derived the equation of flow through unconfined aquifer. The reason for the unsteady flow in an unconfined aquifer may be due to the change in the hydraulic heads with time or compressibility of the mineral grains of the soil matrix forming the aquifer or compressibility of the water stored in the voids within the soil matrix. But in our study, we have assumed the unsteady flow due to the change in the hydraulic heads with time.

The governing equation is the non-linear partial differential equation. The similarity solution is obtained using Lie-infinitesimal transformation technique. Sophus Lie has developed the Lie transformation technique which maps a given differential equation to itself. Such equation remains invariant under some continuous group of transformations which do not alter the structural form of the given equation. Hence similarity solutions can be obtained. The transformation reduces partial differential equation into ordinary differential equation. The analytical solution of resulting equation is obtained.

Mathematical Formulation

In hydrology, the equation of groundwater flow may be used to describe the flow of groundwater through an aquifer. The transient flow of groundwater is described by a form of the diffusion equation while the steady-state flow is described by a form of the Laplace equation. In unconfined aquifers, the solution to the three-dimensional form of the equation is complicated by the presence of a free surface water table boundary condition.

In this paper we have considered the transient flow in one-dimension. Equation for the groundwater flow may be obtained by using the Dupuit–Forchheimer assumption. According to this, it is assumed that hydraulic heads vary only along the horizontal direction i.e along x- direction with time.

For incompressible water in unconfined aquifers, the change in the hydraulic heads with time is equal to the negative divergence of the flux and the source terms, so mathematically it is expressed as

$$S_s \frac{\partial nb}{\partial t} = -\frac{\partial q}{\partial x} - G \tag{1}$$

where n is the porosity of the aquifer, G is the source or sink term which represents the addition of water in the vertical direction (e.g. recharge area), b is the thickness of saturation which is defined as the vertical distance between the water table surface and the aquifer base and S_s is the specific storage.
If we assume the uniform horizontal flow along the entire saturated thickness of the aquifer, using Darcy's law relating flux to hydraulic heads in terms of integrated discharges, we have

$$q = -K\,b\frac{\partial h}{\partial x} \tag{2}$$

where K is the hydraulic conductivity.
Substituting (2) in (1), we obtain

$$S_s \frac{\partial n b}{\partial t} = \frac{\partial}{\partial x}\left(K\,b\frac{\partial h}{\partial x}\right) - G \tag{3}$$

Dividing the above equation by the specific storage , and taking α = K/S_s , the hydraulic diffusivity as unity and taking the unconfined saturated thickness b equal to the hydraulic head h as the hydraulic heads do not vary in the vertical direction and the aquifer base is at the zero datum, and also taking the porosity of the aquifer as unity, we will obtain the governing equation for incompressible saturated groundwater flow as

$$\frac{\partial h}{\partial t} = \frac{\partial}{\partial x}\left(h\frac{\partial h}{\partial x}\right) + N \tag{4}$$

where G is divided by the appropriate storage term and is labeled as N.
Equation (4) is the governing non-linear partial differential equation for unconfined, unsteady and saturated flow in an aquifer.
The boundary conditions are specified as

$$h\left(0,t\right) = 0\,, \qquad \left(0 < t < \infty\right)$$

which shows that the level in aquifer remains fixed at its initial end at any time

$$\frac{\partial h\left(L,t\right)}{\partial x} = 0\,, \qquad \left(0 < t < \infty\right)$$

which shows that the clay embankment at $x = L$ is impermeable

$$h\left(x,0\right) = 0\,, \qquad \left(0 < x < L\right)$$

which gives the initial condition showing that the free surface in the aquifer has the same constant elevation as the level in the aquifer at $t = 0$.

Mathematical Solution

(a) Transforming partial differential equation to ordinary differential equation :

In the present paper, we have applied one- parameter Lie group of transformation technique to find the analytical solution of equation (4). N is the source term and is a dependent function on x and t. We have assumed N to be of the form $A\left(\frac{x}{t}\right)^2$ where A > 0. We have taken the one-parameter infinitesimal Lie-transformation that leaves the surface equation invariant by changing the variables from $\left(x,t\right)$ to $\left(x^*,\ t^*\right)$ as

$$h^{*} = h + \varepsilon\, \eta(x,t,h) + O(\varepsilon^{2})$$
$$t^{*} = t + \varepsilon\, \tau(x,t,h) + O(\varepsilon^{2}) \qquad (5)$$
$$x^{*} = x + \varepsilon\, \xi(x,t,h) + O(\varepsilon^{2})$$

We will find those infinitesimals (ξ, τ, η) for which $h^{*}(x^{*}, t^{*})$ becomes a solution of

$$\frac{\partial h^{*}}{\partial t^{*}} = \frac{\partial}{\partial x^{*}}\left(h^{*}\frac{\partial h^{*}}{\partial x^{*}}\right) + A\left(\frac{x^{*}}{t^{*}}\right)^{2} \qquad (6)$$

whenever $h(x,t)$ is a solution of (4) with $N = A\left(\frac{x}{t}\right)^{2}$.

Substituting the new variables from (5) into equation (6) and applying the classical method of equating to zero the terms with the derivatives of h, we get a set of equations. These set of equations are then solved for (ξ, τ, η) and we finally obtain the values of infinitesimals as

$$\xi = \left(\frac{1}{2}\alpha + \beta\right)x + \gamma \qquad (7)$$
$$\tau = \alpha t + \delta$$
$$\eta = 2h\beta$$

where $\alpha, \beta, \gamma, \delta$ are four arbitrary parameters.

Assuming the invariance of $x = 0$, $t = 0$ and $h(0, t) = 0$, the four parameter group of infinitesimals (7) are reduced to one-parameter group as

$$\xi = \frac{1}{2}\alpha x \qquad (8)$$
$$\tau = \alpha t$$
$$\eta = 0$$

The general partial differential equation of an invariant surface can be given by

$$\xi(x, t, h)\frac{\partial h}{\partial x} + \tau(x, t, h)\frac{\partial h}{\partial t} = \eta(x, t, h) \qquad (9)$$

The characteristic equations corresponding to equation (9) can be written as

$$\frac{dx}{\xi} = \frac{dt}{\tau} = \frac{dh}{\eta} \qquad (10)$$

Substituting values from equation (8), we can obtain the characteristic equations as

$$\frac{dx}{\frac{1}{2}\alpha x} = \frac{dt}{\alpha t} = \frac{dh}{0} \qquad (11)$$

Since $\frac{\xi}{\tau}$ is independent of h, we obtain the similarity form for the solution and the similarity variable $\varsigma(x, t) =$ constant can be obtained from the first equality of (10).

Solving the first equality of equation (11), we get

$$\varsigma = \frac{x}{t^{1/4}} \tag{12}$$

The similarity solution is a dependent function and is given by $h = F(\varsigma).t^{-1/2}$ where the value of ς is as in (12).

Substituting the similarity form of the solution in equation (4), the partial differential equation can be converted into the non-linear ordinary differential equation, given by

$$F'^{2} + F F'' + A\varsigma^{2} + \frac{1}{4}\varsigma F' + \frac{1}{2} F = 0 \tag{13}$$

Equation (13) can be written as

$$\frac{d}{d\varsigma}\left[\varsigma\left(F\frac{dF}{d\varsigma} + \frac{1}{4}F\varsigma + A\frac{\varsigma^{3}}{3}\right) - \frac{F^{2}}{2} - A\frac{\varsigma^{4}}{12}\right] = 0 \tag{14}$$

Integrating both sides, we get

$$F\frac{dF}{d\varsigma}\varsigma + \frac{1}{4}F\varsigma^{2} - \frac{F^{2}}{2} + A\frac{\varsigma^{4}}{4} = c \tag{15}$$

where c acts as the constant of integration.

(b) Solution of non-linear ordinary differential equation :

Equation (15) is Abel's equation of second kind. Comparing it with the general form

$$(g_0 + g_1 y)y' = f_0 + f_1 y + f_2 y^2 + f_3 y^3$$

we will get $g_0 = 0$, $g_1 = \varsigma$, $f_0 = C - \frac{A\varsigma^4}{4}$, $f_1 = -\frac{\varsigma^2}{4}$, $f_2 = \frac{1}{2}$

Taking $\varsigma F = \frac{1}{z}$ in equation (15), it reduces to Abel's equation of first kind

$$z' = -\varsigma\left(C - \frac{A\varsigma^4}{4}\right)z^3 + \frac{\varsigma^2}{4}z^2 - \frac{3}{2\varsigma}z \tag{16}$$

Comparing equation (16) with Abel's equation of first kind

$$u' = f_3 u^3 + f_2 u^2 + f_1 u + f_0$$

we will get $f_0 = 0$, $f_1 = -\frac{3}{2\varsigma}$, $f_2 = \frac{\varsigma^2}{4}$, $f_3 = -\varsigma\left(C - \frac{A\varsigma^4}{4}\right)$

Taking $z = uZ + v$ in equation (16), where u and v are free parameters.

Equation (16) takes the form

$$Z'(\varsigma) = u^2 \; f_3 \left[Z^3 + \phi_1(\varsigma) Z^2 + \phi_2(\varsigma) Z + \phi_3(\varsigma) \right] \qquad (17)$$

We will obtain a system of auxiliary equations with ϕ_1, ϕ_2, ϕ_3 as the free functions.

Assuming that $\phi_1(\varsigma) = \phi_2(\varsigma) = 0$ and $\phi_3(\varsigma) = \varphi(\varsigma)$ we will obtain the values of u and v as

$$u = \exp\left\{ \int \left(f_1 - \frac{f_2^{\,2}}{3 f_3} \right) d\varsigma \right\} = \frac{\left(C - \frac{A\varsigma^4}{4} \right)^{\frac{1}{48A}}}{\varsigma^{\frac{3}{2}}} \quad \text{and} \quad v = -\frac{f_2}{3 f_3} = \frac{\varsigma}{12\left(C - \frac{A\varsigma^4}{4} \right)} \qquad (18)$$

Substituting u and v in equation (17), it will be reduced to

$$Z'(\varsigma) = u^2 \; f_3 \left[Z^3 + \varphi(\varsigma) \right]$$

Taking $\zeta = \int f_3 u^2 \; d\varsigma$ in the above equation, we will get the canonic form for Abel's equation as

$$Z'(\zeta) = Z^3(\zeta) + \varphi(\zeta)$$

Adding and subtracting the term θZ in the canonic form where θ is the free function and solving for Z and φ, we will get

$$Z(\zeta) = \frac{e^{\int \theta d\zeta}}{\sqrt{C - \int e^{2\int \theta d\zeta} \; d\zeta}} \quad \text{and} \quad \varphi(\zeta) = \frac{\theta \; e^{\int \theta \, d\zeta}}{\sqrt{C - \int e^{2\int \theta \, d\zeta} \; d\zeta}}$$

This shows that for the canonic form

$$Z'(\zeta) = Z^3(\zeta) + \frac{\theta \; e^{\int \theta \, d\zeta}}{\sqrt{C - \int e^{2\int \theta \, d\zeta} \; d\zeta}}$$

the solution is given by

$$Z(\zeta) = \frac{e^{\int \theta d\zeta}}{\sqrt{C - \int e^{2\int \theta d\zeta} \; d\zeta}} \qquad (19)$$

Hence the solution of equation (16) is given by

$$z = u Z + v \qquad (20)$$

where the values of u, v and Z are given by equations (18) and (19).

Since $\varsigma F = \frac{1}{z}$, so

$$F = \frac{1}{\varsigma z} \qquad (21)$$

is the solution of equation (15) with the value of z given by equation (20).

Hence the Similarity solution of equation (4) is obtained as $h = F(\varsigma)\, t^{-1/2}$ with $\varsigma = \frac{x}{t^{1/4}}$ and the value of F given by (21).

Conclusion

The main aim of this paper is to find the analytical similarity solution using the method of infinitesimal transformations of partial differential equations. The partial differential equation is reduced to ordinary equation under this transformation. The resulting ordinary differential equation is an Abel's equation of second kind whose analytical solution under certain transformation is obtained.

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