International
Journal
Of Advanced Research in Engineering& Management (IJAREM)
ISSN: 2456-2033 || PP. 01-05

Integral Solutions of Ternary Quadratic Diophantine Equation $3\left(x^2+y^2\right)-5xy=20z^2$

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Abstract: The cone represented by the ternary quadratic Diophantine equation $3\left(x^2+y^2\right)-5xy=20z^2$ is analyzed for its patterns of non-zero distinct integral solutions.
Keywords: Ternary quadratic, cone, integral solutions.
2010 mathematics subject classification: 11D09

1. Introduction:

The Diophantine equation offers an unlimited field for research due to their variety [1-3]. In particular, one may refer [4-14] for quadratic equations with three unknowns. This communication concerns with yet another interesting equation $3\left(x^2+y^2\right)-5xy=20z^2$ representing homogeneous quadratic with three unknowns for determining its infinitely many non-zero integral points.

2. Method of Analysis

The ternary quadratic Diophantine equation to be solved for its non-zero distinct integral solution is

$$3\left(x^2+y^2\right)-5xy=20z^2 \tag{1}$$

Introduction of the linear transformations $(u \neq v \neq 0)$

$$x=u+v\,,\, y=u-v \tag{2}$$

in (1) leads to

$$u^2+11v^2=20z^2 \tag{3}$$

We present below different methods of solving (1).

Pattern - 1:
Write 20 as

$$20=\left(3+i\sqrt{11}\right)\left(3-i\sqrt{11}\right) \tag{4}$$

Assume

$$z=a^2+11b^2 \tag{5}$$

Where a and b are non-zero distinct integers.

Using (4) and (5) in (3), we get

$$u^2+11v^2=\left(3+i\sqrt{11}\right)\left(3-i\sqrt{11}\right)\left(a^2+11b^2\right)^2$$

Employing the method of factorization, the above equation is written as

$$\left(u+i\sqrt{11}v\right)\left(u-i\sqrt{11}v\right)=\left(3+i\sqrt{11}\right)\left(3-i\sqrt{11}\right)\left(a+i\sqrt{11}b\right)^2\left(a-i\sqrt{11}b\right)^2$$

Equating positive and negative factors, the resulting equations are

$$\left(u+i\sqrt{11}v\right)=\left(3+i\sqrt{11}\right)\left(a+i\sqrt{11}b\right)^2 \tag{6}$$

$$\left(u-i\sqrt{11}v\right)=\left(3-i\sqrt{11}\right)\left(a-i\sqrt{11}b\right)^2 \quad (7)$$

Equating real and imaginary parts either in (6) or (7), we get

$$u=3a^2-33b^2-22ab$$
$$v=a^2-11b^2+6ab$$

Substituting the value of u and v in (2)

$$x(a,b)=4a^2-44b^2-16ab \quad (8)$$
$$y(a,b)=2a^2-22b^2-28ab \quad (9)$$

Thus (8), (9) and (5) represent non-zero distinct integral solution of (1) in two parameters.

Note:
It is seen that 20 is also represented as follows:

i) $$20=\frac{\left(13+i\sqrt{11}\right)\left(13-i\sqrt{11}\right)}{9}$$

ii) $$20=\frac{\left(2+i4\sqrt{11}\right)\left(2-i4\sqrt{11}\right)}{9}$$

iii) $$20=\frac{\left(18+i4\sqrt{11}\right)\left(18-i4\sqrt{11}\right)}{25}$$

Following the above procedure, the solutions of (1) for choices (i), (ii) and (iii) are presented below:

Solutions for choice (i)

$$x(A,B)=42A^2-462B^2+12AB$$
$$y(A,B)=36A^2-396B^2-144AB$$
$$z(A,B)=9A^2+99B^2$$

Solutions for choice (ii)

$$x(A,B)=18A^2-198B^2-252AB$$
$$y(A,B)=-6A^2+66B^2-276AB$$
$$z(A,B)=9A^2+99B^2$$

Solutions for choice (iii)

$$x(A,B)=110A^2-1210B^2-260AB$$
$$y(A,B)=70A^2-770B^2-620AB$$
$$z(A,B)=25A^2+275B^2$$

Pattern - 2:
One may write (3) as

$$u^2-9z^2=11\left(z^2-v^2\right)$$
$$(u-3z)(u+3z)=11(z-v)(z+v) \quad (10)$$

Equation (10) is written in the form of ratio as

$$\frac{u+3z}{z+v}=\frac{11(z-v)}{u-3z}=\frac{\alpha}{\beta}\ ,\beta\neq 0 \qquad (11)$$

Which is equivalent to the system of double equations

$$\beta u-\alpha v+z(3\beta-\alpha)=0 \qquad (12)$$

$$\alpha u+11\beta v+z(-3\alpha-11\beta)=0 \qquad (13)$$

Applying the method of cross multiplication for solving (12) and (13)

$$u=3\alpha^2-33\beta^2+22\alpha\beta$$

$$v=-\alpha^2+11\beta^2+6\alpha\beta$$

$$z=z(\alpha,\beta)=\alpha^2+11\beta^2 \qquad (14)$$

Substituting the value of u and v in (2) ,one has

$$\left.\begin{aligned} x=x(\alpha,\beta)=2\alpha^2-22\beta^2+28\alpha\beta,\\ y=y(\alpha,\beta)=4\alpha^2-44\beta^2+16\alpha\beta \end{aligned}\right\} \qquad (15)$$

Thus,(14) and (15) represent the integer solutions to (1).

Note:

Apart from (5) , (10) is also written in the form of ratio as presented below:

i) $\frac{u-3z}{z+v}=\frac{11(z-v)}{u+3z}=\frac{\alpha}{\beta}$

ii) $\frac{u-3z}{11(z-v)}=\frac{z+v}{u+3z}=\frac{\alpha}{\beta}$

iii) $\frac{u+3z}{11(z-v)}=\frac{z+v}{u-3z}=\frac{\alpha}{\beta}$

Following the above procedure, the solutions of (1) for choices (i), (ii) and (iii) are presented below:

Solutions for choice (i)

$$x=x(\alpha,\beta)=-4\alpha^2+44\beta^2+16\alpha\beta$$

$$y=y(\alpha,\beta)=-2\alpha^2+22\beta^2+28\alpha\beta$$

$$z=z(\alpha,\beta)=\alpha^2+11\beta^2$$

Solutions for choice (ii)

$$x=x(\alpha,\beta)=22\alpha^2-2\beta^2-28\alpha\beta$$

$$y=y(\alpha,\beta)=44\alpha^2-4\beta^2-16\alpha\beta$$

$$z=z(\alpha,\beta)=-(11\alpha^2+\beta^2)$$

Solutions for choice (iii)

$$x=x(\alpha,\beta)=-44\alpha^2+4\beta^2-16\alpha\beta$$

$$y=y(\alpha,\beta)=-22\alpha^2+2\beta^2-28\alpha\beta$$

$$z=z(\alpha,\beta)=-(11\alpha^2+\beta^2)$$

Pattern - 3:

Equation (3) is written as

$$u^2 + 11v^2 = 20z^2 * 1 \quad (16)$$

Write 1 as

$$1 = \frac{(5 + i\sqrt{11})(5 - i\sqrt{11})}{36} \quad (17)$$

Using (4), (5) and (17) in (16) ,we get on factorization

$$(u + i\sqrt{11}v)(u - i\sqrt{11}v) = (3 + i\sqrt{11})(3 - i\sqrt{11})(a + i\sqrt{11}b)^2 (a - i\sqrt{11}b)^2 \frac{(5 + i\sqrt{11})(5 - i\sqrt{11})}{36}$$

Equating the positive and negative factors, the resulting equations are,

$$(u + i\sqrt{11}v) = (3 + i\sqrt{11})(a + i\sqrt{11}b)^2 \frac{(5 + i\sqrt{11})}{6}$$

$$(u - i\sqrt{11}v) = (3 - i\sqrt{11})(a - i\sqrt{11}b)^2 \frac{(5 - i\sqrt{11})}{6}$$

Equating real and imaginary parts and replacing a by 3A, b by 3B, we have

$$u = 6A^2 - 66B^2 - 264AB$$

$$v = 12A^2 - 132B^2 + 12AB$$

And from (5), one has

$$z(A,B) = 9A^2 + 99B^2 \quad (18)$$

Substituting the values of u and v in (2), we get

$$\left.\begin{aligned} x(A,B) &= 18A^2 - 198B^2 - 252AB, \\ y(A,B) &= -6A^2 + 66B^2 - 276AB \end{aligned}\right\} \quad (19)$$

Thus, (18) and (19) represent the integer solutions to (1).

Note:

It is seen that 1 on the R.H.S. of (16) is also represented as follows

i) $1 = \frac{(1 + i3\sqrt{11})(1 - i3\sqrt{11})}{100}$

ii) $1 = \frac{(7 + i4\sqrt{11})(7 - i4\sqrt{11})}{225}$

Following the above procedure, the solutions of (1) for choices (i) and (ii) are presented below:

Solutions for choice (i)

$$x(a,b) = -2a^2 + 22b^2 - 28ab$$

$$y(a,b) = -4a^2 + 44b^2 - 16ab$$

$$z(a,b) = a^2 + 11b^2$$

Solutions for choice (ii)

$$x(A,B) = -60A^2 + 660B^2 - 6960AB$$

$$y(A,B) = -630A^2 + 6930B^2 - 5580AB$$
$$z(A,B) = 225A^2 + 2475B^2$$

3. Conclusion

In this paper, we have made an attempt to obtain all integer solutions to (1). As (1) is symmetric in x, y, z it is to be noted that, if (x, y, z) is any positive integer solution to (1),then the triples (-x, y, z), (x, -y, z), (x, y, -z), (x, -y,- z), (-x, y,- z),(-x,- y, z), (-x,- y,- z) also satisfy (1). One may search for integer solutions to other choices of ternary quadratic diophantine equations

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